

The One-Loop Effective Kähler Potential.

I : Chiral Multiplets.

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Abstract

We derive a universal formula for the one-loop renormalization of the effective Kähler potential that applies to general supersymmetric effective field theories of chiral multiplets, with arbitrary interactions respecting $\mathcal{N} = 1$ supersymmetry in four dimensions. The resulting expression depends only on the tree-level mass spectrum and the form of the regulator. This formula simplifies and generalizes existing results in the literature. We include two examples to illustrate its use.

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1. Introduction

The last two decades have seen much progress in the understanding of quantum corrections to supersymmetric field theories with four supercharges. This is mostly attributable to the highly constrained behavior of BPS terms in the effective action – in particular to the holomorphic superpotential, where certain non-renormalization properties have been demonstrated both in perturbative [1] as well as in non-perturbative [2] calculations, and in various contexts in string theory and supergravity (see *e.g.* [3,4,5] and references therein).

On the other hand, the effective Kähler potential generally receives quantum corrections. The perturbative one-loop contribution was the subject of many studies in a variety of cases, as *e.g.* in the superspace calculation of [6] for the Wess-Zumino model, and in [7] for nonlinear sigma models. A more systematic approach was carried out in [8] for renormalizable theories in $D = 4$, and generalized in [9] to the non-renormalizable case, albeit still excluding supersymmetric higher-derivative terms. The importance of the latter was emphasized *e.g.* in [10], both as generic features of the (Wilsonian) low-energy effective action as well as in the context of some supersymmetry breaking scenarios.

The purpose of the present work is to develop a theory of the renormalization of the effective Kähler potential at one loop which is general enough to tackle generic effective field theories and still powerful enough to allow simple calculation even in complicated situations.

We think that there are reasons for which such a general approach is desirable. The Kähler potential can transmit CP- and flavor-breaking effects in realistic candidate models of supersymmetric particle physics that are tiny but nonetheless highly constrained experimentally. Controlling the renormalization of the Kähler potential in effective theories is therefore an essential but often tedious task, which our formula simplifies.

In the hope that ideas along these lines may help winning the reader's approval for a general treatment of the quantum correction ΔK of the Kähler potential, let us proceed by stating our result. We shall show that, for any given regulator, there is a simple formula for ΔK at one loop that reduces to a sum over on-shell degrees of freedom running in the loop, with each massive degree of freedom contributing an amount depending only on its mass-squared. We refer to this property by saying that the formula for the one-loop renormalization of K by virtual degrees of freedom is purely *mass-dependent*. By this we mean to emphasize that none of the three- or four-point couplings affects the one-loop value of ΔK , except through their influence on the physical masses of the linearized modes.

The unfamiliar aspect of the mass-dependent formula is that unphysical degrees of freedom, both positive and negative norm poles of the propagators that lie above the cutoff of the theory, contribute democratically with the physical degrees of freedom whose masses lie below the cutoff. We will show that there is no contradiction here, so long as we are working in a self-consistent regime of the application of the effective field theory.

In the present paper, we restrict ourselves to the case where the degrees of freedom circulating in the loop consist of chiral multiplets only. When massive vector multiplets are present, there is a similar mass-dependent formula. Due to gauge invariance, there are

subtleties involved in its derivation and interpretation, so for clarity of presentation we will defer the discussion of vector multiplets to a subsequent paper. For both chiral and vector multiplets, the contribution of massless multiplets is *not* purely mass-dependent, but in both cases can be derived by taking an $m \rightarrow 0$ limit of the corresponding mass-dependent formulæ for massive degrees of freedom.

The organization of this paper is as follows. After fixing some notation in section 2, we compute in section 3 the one-loop effective Kähler potential for a general effective theory of a single chiral superfield in four dimensions with $\mathcal{N} = 1$ supersymmetry, defined at the quantum level with a general supersymmetric regularization procedure. We then compare the result to the spectrum of linearized solutions to the equations of motion and find that the one-loop correction to the Kähler potential is always of the form

$$\Delta K = \frac{\hbar}{2(2\pi)^D} \int \frac{d^D P}{P^2} \sum_{j=1}^J \left[\ln(P^2 + \mu_j^2) - \ln(\mu_j^2) \right]. \quad (1.1)$$

The sum runs over all supermultiplets of on-shell excitations with masses μ_j , including unphysical super-cutoff solutions to the equations of motion with masses lying far above the cutoff. However, in minimal subtraction schemes such as $\overline{\text{DR}}$ the contributions from the unphysical super-cutoff solutions vanish if equation (1.1) is properly interpreted, and the sum can be restricted to the light modes. In section 4 we show that the validity of equation (1.1) extends to the case of an arbitrary number of chiral superfields. In section 5 we consider examples in four and two dimensions to shed light on the regularization issue. Appendix A summarizes our conventions. Finally, Appendix B is devoted to several useful identities that were used in the derivation of (1.1).

2. Notation for supertraces and superdeterminants

For operators \mathcal{O} acting on the ring of functions on superspace, the supertrace is defined as

$$\text{str}(\mathcal{O}) \equiv \text{tr} \left((-1)^F \mathcal{O} \right) = \text{tr} \left(\mathcal{O} (-1)^F \right), \quad (2.1)$$

where F is the fermion number operator. With the supertrace we define the superdeterminant (Berezinian) in the usual way as

$$\text{sdet}(\mathcal{O}) = \exp\{\text{str}(\ln(\mathcal{O}))\}. \quad (2.2)$$

We shall work with various partial traces whose notation we will now introduce. Our focus is on theories with four real supercharges in $D \leq 4$ so that locally smooth functions on superspace form a space of the structure $C^\infty(\mathbb{R}^D) \otimes \mathcal{V}$, where \mathcal{V} is a 16-dimensional graded vector space (built from all combinations of θ s). Restrictions of traces or determinants over only \mathcal{V} will be denoted by a subscript ($\text{tr}_{\mathcal{V}}$, *etc.*). In particular, the restriction of the supertrace can be written in terms of the integral expression

$$\text{str}_{\mathcal{V}}(\mathcal{O}) = 16 \int d^2\theta d^2\bar{\theta} d^2\kappa d^2\bar{\kappa} \exp\{-\kappa\theta - \bar{\kappa}\bar{\theta}\} \mathcal{O} [\exp\{+\kappa\theta + \bar{\kappa}\bar{\theta}\}], \quad (2.3)$$

where the κ and $\bar{\kappa}$ are Grassmann parameters that play the roles of possible “eigenvalues” of the derivatives D_α and $\bar{D}_{\dot{\alpha}}$. The square brackets mean that \mathcal{O} is evaluated on the exponential contained in the brackets. The demonstration that this formula holds true can be done straightforwardly by choosing a basis for the operators on the 16-dimensional graded vector space.

We will also need to take the supertrace over chiral and antichiral superfields alone¹. For this we define the projection operator

$$\hat{\mathbb{P}} = \begin{pmatrix} \mathbf{P}_\chi & 0 \\ 0 & \mathbf{P}_{\bar{\chi}} \end{pmatrix} , \quad (2.4)$$

where \mathbf{P}_χ and $\mathbf{P}_{\bar{\chi}}$ are the projectors onto chiral and antichiral superfields (A5), (A6), respectively. The “constrained supertrace” over chiral and antichiral superfields shall then be

$$\text{Cstr}(\mathcal{O}) = \text{str}(\hat{\mathbb{P}} \mathcal{O}) = \text{tr}((-1)^F \hat{\mathbb{P}} \mathcal{O}) , \quad (2.5)$$

with the constrained superdeterminant

$$\text{Csdet}(\mathcal{O}) = \exp\{\text{str}(\hat{\mathbb{P}} \ln(\mathcal{O}))\} = \exp\{\text{tr}((-1)^F \hat{\mathbb{P}} \ln(\mathcal{O}))\} . \quad (2.6)$$

In an analogous way we define the constrained *trace* of the operator \mathcal{O} in the ordinary case as

$$\text{Ctr}(\mathcal{O}) \equiv \text{tr}(\hat{\mathbb{P}} \mathcal{O}) , \quad (2.7)$$

and its constrained determinant as

$$\text{Cdet}(\mathcal{O}) = \exp\{\text{ctr}(\ln(\mathcal{O}))\} = \exp\{\text{tr}(\hat{\mathbb{P}} \ln(\mathcal{O}))\} . \quad (2.8)$$

The advantage of writing the supertrace as in (2.3) is that it gives us a quantity which yields the supertrace upon integration over the Grassmann parameters θ and $\bar{\theta}$. We define

$$\text{Istr}_\mathcal{V}(\mathcal{O}) \equiv 16 \int d^2\kappa d^2\bar{\kappa} \exp\{-\kappa\theta - \bar{\kappa}\bar{\theta}\} \mathcal{O} [\exp\{+\kappa\theta + \bar{\kappa}\bar{\theta}\}] . \quad (2.9)$$

In the next section we will consider the particular case of an operator \mathcal{O} which is a superfield, and hence completely defined if we know its value at $\theta = \bar{\theta} = 0$. A straightforward calculation shows that the θ -integrand of the supertrace at $\theta = \bar{\theta} = 0$ is then given by

$$\text{Istr}_{\mathcal{V}|0}(\mathcal{O}) = \mathcal{O} [\theta^2 \bar{\theta}^2]_{\theta=\bar{\theta}=0} . \quad (2.10)$$

For a translationally invariant operator \mathcal{O} we can split the trace into traces over spaces E_P of fixed momentum P and write

$$\begin{aligned} \text{str}(\mathcal{O}) &= \frac{V_D}{(2\pi)^D} \int dE d^{D-1}P \text{str}_{E_P \otimes \mathcal{V}}(\mathcal{O}) \\ &= 16 \frac{V_D}{(2\pi)^D} \int dE d^{D-1}P d^2\theta d^2\bar{\theta} d^2\kappa d^2\bar{\kappa} \exp\{-\kappa\theta - \bar{\kappa}\bar{\theta}\} \mathcal{O}(P) [\exp\{+\kappa\theta + \bar{\kappa}\bar{\theta}\}] \end{aligned} \quad (2.11)$$

¹This is the primary difference between our method and that of [8], where instead of performing this sort of chirally projected path integral, chiral fields were replaced with unconstrained superfields $\Phi = \bar{D}^2\Psi$ (and ghosts that don’t contribute at leading order).

and so on, where $\mathcal{O}(P)$ stands for the operator restricted to the eigenspace of eigenvalue P and V_D is the space-time volume. Throughout the calculation we will assume that the volume and the integral are regularized in some way. In fact, different regularization prescriptions will have different effects eventually. For the sake of clarity we will, however, in the following two sections ignore the ambiguities of regularization. We shall come back to this in section 5.

3. Loop correction from a single chiral superfield

In this section we derive the one-loop effective Kähler potential in the form (1.1) for a general $\mathcal{N} = 1$ supersymmetric effective field theory of a single chiral superfield. We begin by presenting a convenient formula for the one-loop effective Kähler potential in terms of the integrand of a supertrace in subsection 3.1. By relating the result of an explicit evaluation of this formula to the spectrum of tree-level masses, we then derive equation (1.1) in subsection 3.2.

3.1. General remarks

For a given background Φ , Φ^\dagger let \mathcal{O} be the operator defining the quadratic action for fluctuations $\chi \equiv \delta\Phi$, $\chi^\dagger \equiv \delta\Phi^\dagger$,

$$S_{\text{quad}} = \frac{1}{2} \int d^{D+4}z \, (\chi^\dagger, \chi) \, \mathcal{O} \begin{pmatrix} \chi \\ \chi^\dagger \end{pmatrix}. \quad (3.1)$$

In this formula, $d^{D+4}z$ is the measure on superspace, and we have used the fact that for chiral superfields an integral over half of the fermionic coordinates can be written in terms of a full superspace integration after including appropriate projectors. Notice that the operator \mathcal{O} as defined is Hermitian if the action is real. Moreover, since we assume that the underlying theory is supersymmetric, it is constructed only out of the operators \hat{P}, D, \bar{D} , and the background superfields Φ . We will come back to the form of \mathcal{O} more explicitly in section 3.2.

The gaussian path integral over fluctuations χ, χ^\dagger then yields

$$Z_{\text{gaussian}} = \left[\text{Csdet}(\mathcal{O}) \right]^{-\frac{1}{2}}. \quad (3.2)$$

For a partition function Z , the effective action is given by

$$S_{\text{eff}} = -i\hbar \ln(Z), \quad (3.3)$$

and extracting the purely one-loop contribution ΔS to the effective action in the fixed background Φ, Φ^\dagger amounts to including only the determinant over Gaussian fluctuations, such that

$$\Delta S = -i\hbar \ln(Z_{\text{gaussian}}) = \frac{i\hbar}{2} \text{Cstr}(\ln(\mathcal{O})) = \frac{i\hbar}{2} \text{str}(\hat{\mathbb{P}} \ln(\mathcal{O})). \quad (3.4)$$

In order to evaluate the 1-loop corrections to the Kähler potential it is sufficient to consider supersymmetric (and translationally-invariant) background field configurations

$$D\Phi = \bar{D}\Phi^\dagger = 0 \ , \quad (3.5)$$

such that $\Phi = \phi = \text{const.}$ and $\Phi^\dagger = \phi^* = \text{const.}$ In this case, the value of the effective Lagrangian \mathcal{L}_{eff} is independent of position, and we have

$$S_{\text{eff}} = V_D \cdot \mathcal{L}_{\text{eff}} \ . \quad (3.6)$$

In particular, for translationally invariant backgrounds the operator \mathcal{O} on Gaussian fluctuations χ and χ^\dagger is also translationally invariant, and thus the corresponding one-loop contribution to the effective Lagrangian is

$$\Delta\mathcal{L} = \frac{i\hbar}{2V_D} \text{str}(\hat{\mathbb{P}} \ln(\mathcal{O})) \ . \quad (3.7)$$

As \mathcal{O} is translationally invariant, we decompose the trace into a basis of eigenfunctions of \hat{P}_μ with eigenvalues P_μ , and write the formula for the one-loop effective Lagrangian as

$$\Delta\mathcal{L} = \frac{i\hbar}{2(2\pi)^D} \int dE d^{D-1}P \text{str}_{E_P \otimes \mathcal{V}}(\hat{\mathbb{P}} \ln(\mathcal{O})) \ . \quad (3.8)$$

The one-loop contribution $\Delta\mathcal{L}$ to the Lagrangian is given by the contribution ΔK to the Kähler potential ,

$$\Delta\mathcal{L} = \int d^2\theta d^2\bar{\theta} \Delta K, \quad (3.9)$$

which we can now express by means of the integrand of the supertrace as defined in section 2 as

$$\Delta K(\phi, \phi^*) = \frac{i\hbar}{2(2\pi)^D} \int dE d^{D-1}P \text{Istr}_{E_P \otimes \mathcal{V}}(\hat{\mathbb{P}} \ln(\mathcal{O})) \ . \quad (3.10)$$

In the next section we will parametrize the general operator \mathcal{O} and compute the value of $\text{Istr}(\hat{\mathbb{P}} \ln \mathcal{O})$ in terms of that parametrization.

3.2. Parametrization and calculation of the 1-loop Kähler potential

We divide the action for fluctuations $\chi = \delta\Phi, \chi^\dagger = \delta\Phi^\dagger$ into a half-superspace term, its complex conjugate, and the full-superspace term:

$$\mathcal{L} = \frac{1}{2} \int d^2\theta \chi \mathcal{O}_{\text{half}} \chi + \frac{1}{2} \int d^2\bar{\theta} \chi^\dagger \mathcal{O}_{\text{half}}^\dagger \chi^\dagger + \frac{1}{2} \int d^2\theta d^2\bar{\theta} [\chi \mathcal{O}_{\text{full}} \chi^\dagger + \chi^\dagger \mathcal{O}_{\text{full}}^\dagger \chi] \ . \quad (3.11)$$

We are working in a background that satisfies the ansatz (3.5), so in particular the background obeys Lorentz invariance. It is important to notice that as a consequence of Lorentz invariance, hermiticity of \mathcal{O} , and the reality of the action, we lose no generality by taking

$$\begin{aligned} \mathcal{O}_{\text{half}} &= A_1(\square) \ , & \mathcal{O}_{\text{half}}^\dagger &= A_1^*(\square) \ , \\ \mathcal{O}_{\text{full}} &= \mathcal{O}_{\text{full}}^\text{T} = \mathcal{O}_{\text{full}}^* = \mathcal{O}_{\text{full}}^\dagger = A_2(\square) \ . \end{aligned} \quad (3.12)$$

Since the half-superspace integral

$$I \equiv \int d^2\theta \chi \mathcal{O}_{\text{half}} \chi \quad (3.13)$$

can always be written as a full superspace integral

$$I = \int d^2\theta d^2\bar{\theta} \chi \left(-\frac{D^2}{4\Box} \mathcal{O}_{\text{half}} \right) \chi , \quad (3.14)$$

and likewise for antichiral superspace integrals

$$\bar{I} \equiv \int d^2\bar{\theta} \chi^\dagger \mathcal{O}_{\overline{\text{half}}} \chi^\dagger = \int d^2\theta d^2\bar{\theta} \chi^\dagger \left(-\frac{\bar{D}^2}{4\Box} \mathcal{O}_{\overline{\text{half}}} \right) \chi^\dagger , \quad (3.15)$$

the quadratic Lagrangian for fluctuations around the given background is in fact

$$\mathcal{L} = \frac{1}{2} \int d^2\theta d^2\bar{\theta} (\chi^\dagger, \chi) \mathcal{O} \begin{pmatrix} \chi \\ \chi^\dagger \end{pmatrix} \quad (3.16)$$

with

$$\mathcal{O} = \begin{pmatrix} A_2(\Box) & (-\frac{\bar{D}^2}{4\Box}) A_1^*(\Box) \\ (-\frac{D^2}{4\Box}) A_1(\Box) & A_2(\Box) \end{pmatrix} . \quad (3.17)$$

From here on out, we will take the action to be “local” – that is, the operators A_1 and A_2 are polynomials or at least approximable in all respects by polynomials in $\Box = -P^2$. Let us collect a few observations about the operators A_1 and A_2 :

- A_2 is real.
- A_1 need not be real.
- A_2 can have an arbitrary dependence on the constant background fields ϕ, ϕ^* .
- The non-constant terms in A_1 can have arbitrary dependence on background fields ϕ, ϕ^* , while the constant term in A_1 must depend holomorphically on ϕ . The constant term in $A_1(\Box)$ is identified as $W''(\phi)$, where W is the superpotential.

The operator \mathcal{O} depends only on P^2 rather than on P in general, so the Wick rotation is particularly unproblematic; we will simply replace the Lorentzian invariant $-E^2 + P_{\text{spatial}}^2$ with the Euclidean invariant P^2 .

Starting from (3.10), we defer the calculation to the Appendix, and take the results (B9) for the logarithm of \mathcal{O} and formula (A12) for str_V of $\hat{\mathbb{P}}$ in order to write the one-loop correction as

$$\Delta K(\phi, \phi^*) = -\frac{\hbar}{2(2\pi)^D} \int d^D P \text{tr}_{E_P} \left(\frac{1}{\Box} \ln \left(A_2^2 - \frac{|A_1|^2}{\Box} \right) \right) . \quad (3.18)$$

As mentioned above, we will characterize ΔK in terms of a polynomial function of P^2 which we shall call the spectral polynomial $\sigma(P^2)$. It will have the property that its roots lie at $-\mu^2$, where μ is a mass of the tree-level spectrum. Our aim is to prove formula (1.1) by relating $\sigma(P^2)$ to the polynomial appearing inside the logarithm of the momentum integrand in (3.18).

So how do we define, and calculate, the spectral polynomial? We would like the roots of $\sigma(P^2)$ to encode the linearized solutions to the equations of motion of the effective field theory, with the correct multiplicities. Written in superspace, these equations are

$$\mathcal{O} \begin{pmatrix} \chi \\ \chi^\dagger \end{pmatrix} = 0, \quad (3.19)$$

and we can solve them on each eigenspace of P separately. The tree-level excitations therefore lie exactly at the values of P^2 such that \mathcal{O} has a vanishing eigenvalue at these specific momenta – that is, when its constrained determinant vanishes on the eigenspace E_P . The constrained determinant \mathbf{Cdet} rather than the ordinary determinant is the relevant object, because propagating modes of χ (χ^\dagger) must satisfy both the equations of motion and the chiral (antichiral) constraint. Thus our definition for the spectral polynomial is

$$\sigma(P^2) := \mathbf{Cdet}_{E_P}(\sqrt{\square} \mathcal{O}). \quad (3.20)$$

The square root of \square has been included in order to remove possible divergences if P goes to 0.

Given our parametrization (3.17), let us write the operator $\sigma(\square)$ in terms of the functions $A_1(\square)$ and $A_2(\square)$. In a derivation which we defer to Appendix B, we obtain in (B12) the expression

$$\sigma(\square) = (\square)^4 \left(A_2(\square)^2 - \frac{|A_1|^2}{\square} \right)^4. \quad (3.21)$$

Hence $\frac{1}{4}\ln(\sigma(\square))$ is the same, up to constant terms, as the operator in the trace which appears in the integrand in (3.18) :

$$\ln \left(A_2^2 - \frac{|A_1|^2}{\square} \right) = -\ln(\square) + \frac{1}{4}\ln(\sigma(\square)). \quad (3.22)$$

Let us consider how this expression is related to the spectrum of linearized modes. The function σ is a polynomial in P^2 of some finite order $4J$, so we can write it in a factorized form as

$$\sigma(P^2) = N \prod_{j=1}^J (P^2 + \mu_j^2)^4. \quad (3.23)$$

Here the μ_j are mass parameters of the linearized modes, and N is a normalization constant. The parameter j counts the distinct multiplets, which in an $\mathcal{N} = 1$ theory form degenerate sets containing four states each. Both the μ_j and N depend on the background fields ϕ and ϕ^* . Let us point out that j in general runs over both “physical states”, *i.e.* linearized states

within the validity regime of the original Lagrangian below some cut-off scale, as well as “super-cut-off states” which lie beyond that scale. Evaluating (3.23) at $P = 0$ leads to the identity

$$N^{\frac{1}{4}} \cdot \prod_{j=1}^J \mu_j^2 = \sigma(0)^{\frac{1}{4}} = |A_1|^2 \Big|_{P^2=0}. \quad (3.24)$$

Now, as we have observed, $A_1|_{P^2=0}$ is the second derivative of the superpotential. Therefore (3.24) is the norm square of a function which is holomorphic in the background field ϕ . If we use (3.24) in (3.23), we therefore find for the logarithm of $\sigma(P^2)$

$$\begin{aligned} \frac{1}{4} \ln(\sigma(P^2)) &= \frac{1}{4} \ln(N) + \sum_{j=1}^J \ln(P^2 + \mu_j^2) \\ &= \sum_{j=1}^J \left[\ln(P^2 + \mu_j^2) - \ln(\mu_j^2) \right] \\ &\quad + (\text{HOLOMORPHIC}) + (\text{ANTIHOLOMORPHIC}), \end{aligned} \quad (3.25)$$

and (3.22) hence yields the expression

$$\begin{aligned} \ln \left(A_2^2 - \frac{|A_1|^2}{\square} \right) \Big|_{\square=-P^2} &= -\ln(-P^2) + \sum_{j=1}^J \left[\ln(P^2 + \mu_j^2) - \ln(\mu_j^2) \right] \\ &\quad + (\text{HOLOMORPHIC}) + (\text{ANTIHOLOMORPHIC}) \end{aligned} \quad (3.26)$$

for the integrand in (3.18). Since holomorphic and constant terms in the Kähler potential do not contribute to the physical action and can be ignored, the final formula for ΔK is

$$\Delta K = \frac{\hbar}{2(2\pi)^D} \int \frac{d^D P}{P^2} \sum_{j=1}^J \left[\ln(P^2 + \mu_j^2) - \ln(\mu_j^2) \right]. \quad (3.27)$$

We point out again that while the integral in this formula is regularized, the sum over j includes both physical and super-cut-off modes at the level of the integrand. Different regularization procedures interact with this feature in different ways. We will come back to this issue by considering concrete examples in section 5.

4. Loop correction from multiple chiral superfields

We now work with n chiral superfields Φ^a , $a = 1, \dots, n$. Expanding again around a given vacuum, we collect the fluctuation fields χ^a into a vector χ . Then the matrix defining the quadratic action

$$\mathcal{L} = \frac{1}{2} \int d^2\theta d^2\bar{\theta} (\chi^\dagger, \chi) \mathcal{O} \begin{pmatrix} \chi \\ \chi^\dagger \end{pmatrix} \quad (4.1)$$

is

$$\mathcal{O} = \begin{pmatrix} \mathcal{O}_{\text{full}} & (-\frac{\bar{D}^2}{4\Box}) \mathcal{O}_{\text{half}} \\ (-\frac{D^2}{4\Box}) \mathcal{O}_{\text{half}} & \mathcal{O}_{\text{full}} \end{pmatrix}, \quad (4.2)$$

where now $\mathcal{O}_{\text{full}}$, $\mathcal{O}_{\text{full}}$, $\mathcal{O}_{\text{half}}$ and $\mathcal{O}_{\text{half}}$ are $n \times n$ matrices. The reality of the action again imposes a hermiticity condition on the matrices,

$$\mathcal{O}_{\text{full}}^\dagger = \mathcal{O}_{\text{full}}, \quad \mathcal{O}_{\text{full}}^\dagger = \mathcal{O}_{\text{full}}, \quad \mathcal{O}_{\text{half}}^\dagger = \mathcal{O}_{\text{half}}, \quad (4.3)$$

and the Bose statistics of the chiral superfields allow us to impose

$$\mathcal{O}_{\text{full}}^T = \mathcal{O}_{\text{full}}, \quad \mathcal{O}_{\text{half}}^T = \mathcal{O}_{\text{half}}, \quad \mathcal{O}_{\text{half}}^T = \mathcal{O}_{\text{half}}. \quad (4.4)$$

Taking also Lorentz invariance into account, we can parametrize our matrix \mathcal{O} as

$$\mathcal{O} \equiv \begin{pmatrix} A_2(\Box) & -\frac{\bar{D}^2}{4\Box} A_1^*(\Box) \\ -\frac{D^2}{4\Box} A_1(\Box) & A_2^*(\Box) \end{pmatrix}, \quad (4.5)$$

where A_1 is symmetric ($A_1^T = A_1$), and A_2 is hermitian. With the ansatz (3.5), we have

$$\Delta K(\phi, \phi^*) = -\frac{\hbar}{2(2\pi)^D} \int d^D P \text{Istr}_{EP \otimes \mathcal{V}} (\hat{\mathbb{P}} \ln(\mathcal{O})) \quad (4.6)$$

in Euclidean signature, where now the supertrace includes a trace over the $n \times n$ matrix structure acting on the n chiral superfields and their conjugates. Since it is constructed from superfields, the integrand is again a superfield as well, and hence determined by its $\theta = \bar{\theta} = 0$ component. We will now proceed in a slightly different way than in section 3.2 and represent Istr , evaluated at $\theta = \bar{\theta} = 0$, as a trace, by using an additional operator insertion. Indeed, for all operators on superspace of the form (4.5), we can write

$$\text{Istr}_{EP \otimes \mathcal{V}|0} (\hat{\mathbb{P}} \ln \mathcal{O}) = \text{tr}_{EP \otimes \mathcal{V}|0} (\mathcal{F} \cdot \hat{\mathbb{P}} \ln \mathcal{O}), \quad (4.7)$$

$$\text{where } \mathcal{F} \equiv \frac{1}{\Box} \left(\frac{1}{2} \mathbf{P}_\chi + \frac{1}{2} \mathbf{P}_{\bar{\chi}} - \frac{1}{4} \cdot \mathbf{1} \right). \quad (4.8)$$

It is not difficult to deduce the form of the operator \mathcal{F} . It is an R-neutral and Lorentz-scalar operator, and taking the trace is an R-neutral and Lorentz-scalar operation, as is taking the integrand of the supertrace and evaluating at zero. On any eigenspace of P_μ , there are exactly sixteen linearly independent operators anticommuting with the Q_α and the $\bar{Q}_{\dot{\alpha}}$ which consist of all the operators composed of D_α and $\bar{D}_{\dot{\alpha}}$ that treat P_μ as a c -number. Of those sixteen operators, only five are Lorentz-scalar, and of those only three are neutral under the R-symmetry: 1 , \mathbf{P}_χ and $\mathbf{P}_{\bar{\chi}}$. So combinations of these are the only operators that we can write as arguments for Istr at $\theta = \bar{\theta} = 0$ for fixed momentum, and they are also the only operators that can have nonvanishing traces with \mathcal{F} . There remain only three undetermined coefficients to fix, and we fixed them.

Since \mathcal{A} commutes with $\hat{\mathbb{P}}$ and all the operators that contribute to \mathcal{O} – i.e. $P^2 = -\square$, \mathbf{P}_χ , $\mathbf{P}_{\bar{\chi}}$, and the identity – we obtain

$$\begin{aligned}
\text{Istr}_{E_P \otimes \mathcal{V}|0}(\hat{\mathbb{P}} \ln(\mathcal{O})) &= \text{tr}_{E_P \otimes \mathcal{V}|0}(\hat{\mathbb{P}} \mathcal{A} \ln(\mathcal{O})) \\
&= \text{tr}_{E_P \otimes \mathcal{V}|0} \left(\hat{\mathbb{P}} \frac{1}{\square} \left(\frac{1}{2} \mathbf{P}_\chi + \frac{1}{2} \mathbf{P}_{\bar{\chi}} - \frac{1}{4} \cdot \mathbf{1} \right) \ln(\mathcal{O}) \right) \\
&= \text{tr}_{E_P \otimes \mathcal{V}|0} \left(\frac{1}{4} \frac{1}{\square} \hat{\mathbb{P}} \ln(\mathcal{O}) \right) \\
&= -\frac{1}{4P^2} \ln \text{Cdet}_{E_P \otimes \mathcal{V}|0}(\mathcal{O}) .
\end{aligned} \tag{4.9}$$

We define the spectral polynomial analogously to the single-field case (3.20) as

$$\sigma(P^2) \equiv \text{Cdet}_{E_P \otimes \mathcal{V}} \left(\sqrt{\square} \mathcal{O} \right) = (-P^2)^{4n} \text{Cdet}_{E_P \otimes \mathcal{V}}(\mathcal{O}) . \tag{4.10}$$

The relation between $\text{Istr}(\hat{\mathbb{P}} \ln(\mathcal{O}))$ evaluated at $\theta = \bar{\theta} = 0$ and the spectral polynomial is then

$$\text{Istr}_{E_P \otimes \mathcal{V}|0}(\hat{\mathbb{P}} \ln(\mathcal{O})) = -\frac{1}{4P^2} \ln(\sigma(P^2)) + \frac{n}{P^2} \ln(-P^2) . \tag{4.11}$$

Since the spectral polynomial is again of the form (3.23) with

$$N^{\frac{1}{4}} \cdot \prod_{j=1}^J \mu_j^2 = \sigma(0)^{\frac{1}{4}} = |\det_{n \times n}(A_1)|^2 \Big|_{P^2=0} , \tag{4.12}$$

following from equation (B24), this shows that in the case of several chiral superfields the one-loop effective Kähler potential also takes the form (1.1).

5. Examples

In this section we apply our formula in certain examples with the aim of illuminating the regularization-dependence of the formula, and in particular the sense in which the formula decouples the unphysical super-cutoff modes consistently despite treating them on a democratic footing with the physical modes, at the level of the momentum integrand.

5.1. Higher derivative example

Computing the one-loop Kähler potential with the mass-dependent formula

Consider a simple Wess-Zumino model with a \square/M^2 type operator. Let the Lagrangian to quadratic order in fluctuations be

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \chi^\dagger \left(1 - \frac{\square}{|M|^2} \right) \chi + \int d^2\theta \frac{1}{2} \chi W'' \chi + \int d^2\bar{\theta} \frac{1}{2} \chi^\dagger \overline{W}'' \chi^\dagger \tag{5.1}$$

From (3.18) with $A_2 = 1 - \square/|M|^2$ and $A_1 = W''$, we can immediately write down the one-loop correction to the Kähler potential.

$$\Delta K = \frac{\hbar}{2(2\pi)^4} \int d^4 P \frac{1}{P^2} \ln \left[\left(1 + \frac{P^2}{|M|^2} \right)^2 + \frac{|W''|^2}{P^2} \right]. \quad (5.2)$$

It is perhaps not obvious that the zeroes of the argument of the log still give masses, but this becomes apparent upon considering the equations of motion,

$$\left(1 - \frac{\square}{|M|^2} \right) \chi = \frac{\overline{W}'' \bar{D}^2}{4\square} \chi^\dagger. \quad (5.3)$$

By acting with D^2 on (5.3) and combining the result with the conjugate equation, we find

$$\left(1 - \frac{\square}{|M|^2} \right)^2 = \frac{|W''|^2}{\square}, \quad (5.4)$$

the solutions of which are

$$\mu_L^2 = |W''|^2 \left(1 + 2 \left| \frac{W''}{M} \right|^2 + 7 \left| \frac{W''}{M} \right|^4 + \mathcal{O} \left(\left| \frac{W''}{M} \right|^6 \right) \right), \quad (5.5)$$

$$\mu_{H\pm}^2 = |M|^2 \left(1 \pm \left| \frac{W''}{M} \right| - \frac{1}{2} \left| \frac{W''}{M} \right|^2 + \mathcal{O} \left(\left| \frac{W''}{M} \right|^3 \right) \right). \quad (5.6)$$

Direct computation by expanding the action in $\frac{1}{M^2}$

For comparison, consider now the calculation of the (1-PI) effective action for a model with action

$$S[\Phi, \Phi^\dagger] = \int d^8 z \Phi^\dagger \left(1 - \frac{\square}{|M|^2} \right) \Phi + \int d^6 z W(\Phi) + \int d^6 \bar{z} \overline{W}(\Phi^\dagger) \quad (5.7)$$

in dimensional regularization to subleading order in $1/M$. The action as written is assumed to define an effective field theory that describes the light degrees of freedom below some matching scale $\mu < M$. One can then simply treat the higher derivative term perturbatively.

The goal will then be to extract the correction to the Kähler potential from

$$\Gamma[\Phi, \Phi^\dagger] = S[\Phi, \Phi^\dagger] - i \ln \left[\int \mathcal{D}\chi \mathcal{D}\chi^\dagger \exp \left\{ i \int d^8 z \chi^\dagger \left(1 - \frac{\square}{|M|^2} \right) \chi + i \int d^6 z \frac{1}{2} \chi W'' \chi + i \int d^6 \bar{z} \frac{1}{2} \chi^\dagger \overline{W}'' \chi^\dagger \right\} \right], \quad (5.8)$$

treating the higher derivative term as a perturbation. This gives

$$\begin{aligned} \Gamma[\Phi, \Phi^\dagger] = S[\Phi, \Phi^\dagger] + \frac{i}{2} \text{str} \hat{\mathbb{P}} \ln \mathcal{O} - i \left[-i \int d^8 z \langle \chi^\dagger(z) \left(\frac{\square}{|M|^2} \right) \chi(z) \rangle_c \right. \\ \left. + \frac{(-i)^2}{2} \int d^8 z \int d^8 z' \langle \chi^\dagger(z) \left(\frac{\square}{|M|^2} \right) \chi(z) \chi^\dagger(z') \left(\frac{\square}{|M|^2} \right) \chi(z') \rangle_c + \dots \right] \end{aligned} \quad (5.9)$$

where \mathcal{O} has $A_1 = W''(\Phi)$ and $A_2 = 1$ and the subscript indicates that only the connected contributions are to be kept. To extract the correction to the Kähler potential, as before we evaluate at $\Phi = \phi$, $\Phi^\dagger = \phi^*$. To subleading order in the interaction, we then have

$$\begin{aligned}\Delta K(\phi, \phi^*) &= \frac{i}{2V} \text{Istr} \hat{\mathbb{P}} \ln \mathcal{O}|_{\theta=\bar{\theta}=0} \\ &- \frac{1}{|M|^2} \int d^8 z' \delta^8(z - z')|_{\theta=\bar{\theta}=0} \square \langle \chi(z) \chi^\dagger(z') \rangle|_{\theta=\bar{\theta}=0} \\ &+ \frac{i}{2|M|^4} \int d^8 z' \square \langle \chi(z) \chi^\dagger(z') \rangle|_{\theta=\bar{\theta}=0} \square \langle \chi(z') \chi^\dagger(z) \rangle|_{\theta=\bar{\theta}=0} \\ &+ \frac{i}{2|M|^4} \int d^8 z' \square \square' \langle \chi(z) \chi(z') \rangle|_{\theta=\bar{\theta}=0} \langle \chi^\dagger(z') \chi^\dagger(z) \rangle|_{\theta=\bar{\theta}=0} \\ &+ \dots\end{aligned}\tag{5.10}$$

The first term is the usual correction for the renormalizable theory. So we focus on corrections suppressed by M . Using the explicit form for the propagator in this background

$$\begin{aligned}\langle \chi(z) \chi^\dagger(z') \rangle &= i \frac{\overline{D}^2 D^2}{16(\square - |m|^2)} \delta^8(z - z'), \\ \langle \chi(z) \chi(z') \rangle &= i \frac{m^* \overline{D}^2}{4(\square - |m|^2)} \delta^8(z - z'), \\ \langle \chi^\dagger(z) \chi^\dagger(z') \rangle &= i \frac{m D^2}{4(\square - |m|^2)} \delta^8(z - z'),\end{aligned}\tag{5.11}$$

where $m = W''(\phi)$, one finds (after continuation to Euclidean momenta)

$$\begin{aligned}\Delta K(\phi, \phi^*) &\supset \frac{1}{|M|^2} \int \frac{d^4 P}{(2\pi)^4} \left(1 - \frac{|m|^2}{P^2 + |m|^2} \right) \\ &- \frac{1}{2|M|^4} \int \frac{d^4 P}{(2\pi)^4} \frac{P^6}{(P^2 + |m|^2)^2} \\ &+ \frac{1}{2|M|^4} \int \frac{d^4 P}{(2\pi)^4} \frac{|m|^2 P^4}{(P^2 + |m|^2)^2} \\ &+ \dots\end{aligned}\tag{5.12}$$

Performing the integral in $4 - 2\epsilon$ dimensions and working with the $\overline{\text{DR}}$ -scheme, we find

$$\Delta K(\phi, \phi^*) \supset \frac{|m|^4(1 - \ln |m|^2/\mu^2)}{16\pi^2|M|^2} + \frac{|m|^6(5 - 7 \ln |m|^2/\mu^2)}{32\pi^2|M|^4} + \dots\tag{5.13}$$

Recalling that $m = W''(\phi)$, we finally have

$$\Delta K(\Phi, \Phi^\dagger) \supset \frac{|W''(\Phi)|^4(1 - \ln |W''(\Phi)|^2/\mu^2)}{16\pi^2|M|^2} + \frac{|W''(\Phi)|^6(5 - 7 \ln |W''(\Phi)|^2/\mu^2)}{32\pi^2|M|^4} + \dots\tag{5.14}$$

This agrees with our formula provided only the light mode is kept. Performing the integrals with a Wilsonian cut-off instead, one finds agreement with the formula if all modes are kept. The reason the heavy modes do not contribute to the final answer in perturbation theory in dimensional regularization can be traced to the fact that the large M expansion of their logarithms leads to scale-free integrals.

That is, in dimensional regularization the formula treats the unphysical super-cutoff and physical modes on a democratic footing at the level of the integrand prior to expanding in the heavy scale M . It is only this expansion that breaks the symmetry among modes that is present until this point. For a Wilsonian regulator, physical and super-cutoff modes both make nonzero contributions to the momentum integrand, but the momentum-dependent contributions of the supercutoff modes are small, with a power of M^2 appearing automatically in the denominator for each power of P^2 in the numerator. Supercutoff modes thus contribute to the integral with powers Λ^2/M^2 , since our domain of integration is restricted to $|P| < \Lambda$.

For a Wilsonian cutoff, the processes of expanding the action, expanding the integrand, and doing the integral, all commute with each other. For dimensional regularization, the first two commute and the third does not. Since the first two processes commute for either regulator, terms of order $M^{-(m+1)}$ and smaller in the tree-level action, manifestly have no effect on the one-loop value of ΔK up to order M^{-m} .

5.2. The two-dimensional sigma model

Let us now check our formula against the known expression [11] for the β -function in the case of two-dimensional sigma models. Although our formula depends only on physical masses, we shall find that by taking a limit as the masses go to zero, we recover the correct β -function governing the anomalous scale dependence of the massless limit.

In going from four to two space-time dimensions, $\mathcal{N} = 1$ supersymmetry reduces to $(2,2)$ supersymmetry. A Kähler potential which is a general polynomial in chiral fields yields a two-dimensional $\mathcal{N} = (2,2)$ sigma model where the scalar components of the fields parametrize a Kähler manifold. For the action

$$S = \int d^2x d^4\theta K(\Phi^i, \Phi^{\dagger\bar{j}}), \quad (5.15)$$

we verify that (1.1) reproduces the well-known fact that the one-loop beta function of the Kähler metric $K_{i\bar{j}} = \partial/\partial\phi^i \partial/\partial\phi^{\bar{j}*} K$ as a coupling is proportional to the Ricci curvature [11]. We shall do this in a Wilsonian regularization scheme, where for a momentum-space UV cut-off Λ

$$\beta^{K_{i\bar{j}}} \equiv -\Lambda \frac{\delta}{\delta\Lambda} K_{i\bar{j}}(\Lambda). \quad (5.16)$$

As (1.1) holds in a theory with massive linearized quantum fluctuations, we must first add masses to our sigma model. This is done by adding a superpotential term to the action (5.15),

$$S = \int d^2x d^4\theta K(\Phi^i, \Phi^{\dagger\bar{j}}) + \left[\int d^2x d^2\theta W(\Phi^i) + \text{h.c.} \right], \quad (5.17)$$

and computing the mass-squared matrix of the linearized modes in the background field formalism, before eventually taking the zero-mass limit. Hence we apply the splitting (3.5) and consider the quadratic Lagrangian. Integrating out the auxiliary components of the fluctuations and performing the integration over the Grassmann parameters leaves us with

$$\mathcal{L} = K_{i\bar{j}} \left[\partial_\mu \chi_0^i \partial^\mu \bar{\chi}_0^{\bar{j}*} - K^{\bar{k}i} \bar{W}_{\bar{k}\bar{l}} K^{\bar{j}m} W_{mn} \bar{\chi}_0^{\bar{l}*} \chi_0^m \right] + \dots, \quad (5.18)$$

where χ_0 (χ_0^*) are the scalar components of the fluctuation superfields χ ($\bar{\chi}$), $K_{i\bar{j}} \equiv K_{i\bar{j}}(\phi, \phi^*)$ is the Kähler potential evaluated on the constant background fields (with inverse matrix $K^{\bar{j}i}$), and $W_{ij} = \partial/\partial\phi^i \partial/\partial\phi^j W(\phi)$ are the second derivatives of the superpotential evaluated in the background. The omitted terms contain only the fermionic components of the fluctuations, such that we can read off the mass-squared matrix from the equations of motion for the χ_0^i as obtained from (5.18)

$$\partial^\mu \partial_\mu \chi_0^i = -M_j^i \chi_0^j, \quad (5.19)$$

with

$$M_j^i = K^{\bar{k}i} \bar{W}_{\bar{k}\bar{l}} K^{\bar{j}m} W_{mj} \equiv (\mathbf{K}^{-1})^T \bar{\mathbf{W}} \mathbf{K}^{-1} \mathbf{W}, \quad (5.20)$$

in an obvious notation. The hermitian conjugate fields lead an analogous matrix which we combine with (5.20) into one mass-squared matrix M_{IJ}^I , where I and J run over both i and \bar{j} indices. For the process of changing our cut-off from Λ to $\Lambda - \Delta\Lambda$ by integrating out modes, (1.1) yields

$$\begin{aligned} \Delta K &= \frac{1}{4\pi} \int_\Lambda^{\Lambda-\Delta\Lambda} \frac{dP}{P} \ln \left[\prod_I \frac{P^2 + \mu_I^2}{\mu_I^2} \right] \\ &= \frac{\Delta\Lambda}{4\pi\Lambda} \left[\ln \left(\prod_I \mu_I^2 \right) - \ln \left(\prod_I (\Lambda^2 + \mu_I^2) \right) \right] + \dots, \end{aligned} \quad (5.21)$$

where we have set $\hbar = 1$ and omitted higher-order terms in $\Delta\Lambda$ in the second line. The μ_I^2 are the eigenvalues of the mass-squared matrix \mathbf{M} , which we can split according to (5.20) such that we obtain

$$\begin{aligned} \Delta K &= \frac{\Delta\Lambda}{4\pi\Lambda} \left[\ln \det \mathbf{M} - \ln \det(\Lambda^2 \cdot \mathbf{1} + \mathbf{M}) \right] \\ &= -\frac{\Delta\Lambda}{4\pi\Lambda} \left[\ln \det |\mathbf{K}|^2 + \ln \det(\mathbf{1} + \Lambda^{-2} \mathbf{M}) \right. \\ &\quad \left. - \ln \det \mathbf{W} - \ln \det \bar{\mathbf{W}} + \ln(\Lambda^2) \right], \end{aligned} \quad (5.22)$$

again up to higher orders in $\Delta\Lambda$. The last three terms in the square bracket of the second equation are either holomorphic or antiholomorphic and can be discarded. If we then take the superpotential to zero, the mass-squared matrix vanishes and the remaining terms have a smooth limit. By taking two derivatives with respect to the zero modes of the scalar fields and using an identity in Kähler geometry,

$$R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \ln \det \mathbf{K}, \quad (5.23)$$

we obtain the familiar result for the one-loop β -function for the supersymmetric nonlinear sigma model:

$$\beta^{K_{i\bar{j}}} = \frac{1}{2\pi} \frac{\partial}{\partial \phi^i} \frac{\partial}{\partial \phi^{\bar{j}*}} \ln \det \mathbf{K} = -\frac{1}{2\pi} R_{i\bar{j}}. \quad (5.24)$$

6. Conclusions and Outlook

In this paper we have proven a universal formula for the one-loop renormalization of the Kähler potential due to chiral multiplets circulating in the loop. The formula generalizes previous results [8],[9] in that it applies to general effective actions with arbitrary numbers of derivatives. It would be useful to see to what extent a similar mass-dependent formula applies to the contribution to the one-loop Kähler potential of multiplets other than chiral multiplets. In particular, the contributions of massive vector multiplets are relevant for the study of realistic theories. For vector multiplets, there are delicate questions of gauge dependence of the Kähler potential off the D-flat moduli space, for which reason one must be careful about defining the effective Kähler potential in a gauge-invariant way. We remark here that a straightforward calculation can be done in unitary gauge [12], precisely in parallel with the calculation presented in this paper, with the result that the massive vector multiplet contributes precisely in the same form as a chiral multiplet of the same mass, with an overall factor of -2. This value is in manifest agreement with [8] in the renormalizable case, and somewhat nontrivially in agreement with [9] in the non-renormalizable two-derivative case.

Four-dimensional renormalizable theories are formulated in terms of chiral and vector multiplets alone, but since our formula applies to effective field theories with a finite cutoff, there is nothing in principle to obstruct extending the analysis presented here to theories with massive higher-spin multiplets. To extend the formula to perturbative but non-Lagrangian theories such as superstring theory would also be interesting, and potentially of value in model building.

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Appendices

A. Superspace conventions

- In all our text, $\square \equiv -\partial_t^2 + \partial_i^2 = \eta^{\mu\nu} \partial_\mu \partial_\nu$, where $\eta^{\mu\nu}$ is the mostly-plus metric.
- For $N = 1$ superspace coordinates $(x, \theta, \bar{\theta})$ in $D = 4$ dimensions we have

$$\begin{aligned} Q &= \frac{\partial}{\partial \theta} - i\sigma^m \bar{\theta} \partial_m, & \bar{Q} &= -\frac{\partial}{\partial \bar{\theta}} + i\theta \sigma^m \partial_m, \\ D &= \frac{\partial}{\partial \theta} + i\sigma^m \bar{\theta} \partial_m, & \bar{D} &= -\frac{\partial}{\partial \bar{\theta}} - i\theta \sigma^m \partial_m, \end{aligned} \quad (\text{A1})$$

where ∂_m is a spatial derivative, σ^m is the vector consisting of the Pauli matrices and the identity, and spinor indices $\alpha, \dot{\alpha}$ are suppressed. If we apply these operators on a chiral superfield, we sometimes implicitly assume that the space-time coordinates have been shifted $x \mapsto x + i\theta\sigma\bar{\theta}$, in which case the expressions are rather

$$\begin{aligned} Q &= \frac{\partial}{\partial \theta}, & \bar{Q} &= -\frac{\partial}{\partial \bar{\theta}} + 2i\theta \sigma^m \partial_m, \\ D &= \frac{\partial}{\partial \theta} + 2i\sigma^m \bar{\theta} \partial_m, & \bar{D} &= -\frac{\partial}{\partial \bar{\theta}}. \end{aligned} \quad (\text{A2})$$

The derivatives satisfy in particular the identities

$$D^2 \bar{D}^2 D^2 = +16 \square D^2, \quad \bar{D}^2 D^2 \bar{D}^2 = +16 \square \bar{D}^2, \quad (\text{A3})$$

and

$$\text{Istr}_0(\bar{D}^2 D^2) = \text{Istr}_0(D^2 \bar{D}^2) = 16. \quad (\text{A4})$$

- The operator

$$\mathbf{P}_\chi \equiv \frac{\bar{D}^2 D^2}{16 \square}, \quad \mathbf{P}_\chi^2 = \mathbf{P}_\chi, \quad (\text{A5})$$

is a projection onto functions on superspace satisfying the chiral constraint. Likewise

$$\mathbf{P}_{\bar{\chi}} \equiv \frac{D^2 \bar{D}^2}{16 \square}, \quad \mathbf{P}_{\bar{\chi}}^2 = \mathbf{P}_{\bar{\chi}}, \quad (\text{A6})$$

is a projection operator onto functions on superspace satisfying the antichiral constraint. These two projectors are mutually excluding,

$$\mathbf{P}_\chi \mathbf{P}_{\bar{\chi}} = \mathbf{P}_{\bar{\chi}} \mathbf{P}_\chi = 0, \quad (\text{A7})$$

but not complementary,

$$\mathbf{P}_s \equiv \mathbf{P}_\chi + \mathbf{P}_{\bar{\chi}} \neq 1. \quad (\text{A8})$$

We define the complement of \mathbf{P}_s to be

$$\mathbf{P}_\perp \equiv 1 - \mathbf{P}_s, \quad (\text{A9})$$

which projects onto the “transverse part” or “gauge invariant part” of a real superfield. For the projectors $\mathbf{P}_\chi, \mathbf{P}_{\bar{\chi}}$ and the derivatives D, \bar{D} we have the following identities:

$$\begin{aligned} \mathbf{P}_\chi D^2 &= \bar{D}^2 \mathbf{P}_\chi = \mathbf{P}_{\bar{\chi}} \bar{D}^2 = D^2 \mathbf{P}_{\bar{\chi}} = 0, \\ \mathbf{P}_{\bar{\chi}} D^2 &= D^2 \mathbf{P}_\chi = D^2, \quad \mathbf{P}_\chi \bar{D}^2 = \bar{D}^2 \mathbf{P}_{\bar{\chi}} = \bar{D}^2. \end{aligned} \quad (\text{A10})$$

Furthermore, we have

$$\text{tr}_\mathcal{V}(\mathbf{P}_\chi) = \text{tr}_\mathcal{V}(\mathbf{P}_{\bar{\chi}}) = 4, \quad \text{tr}_\mathcal{V}(\mathbf{P}_s) = \text{tr}_\mathcal{V}(\mathbf{P}_\perp) = 8, \quad (\text{A11})$$

and

$$\text{Istr}_{\mathcal{V}|0}(\mathbf{P}_\chi) = \text{Istr}_{\mathcal{V}|0}(\mathbf{P}_{\bar{\chi}}) = \frac{1}{\square}, \quad \text{Istr}_{\mathcal{V}|0}(\hat{\mathbb{P}}) = \frac{2}{\square}. \quad (\text{A12})$$

B. The kinetic operator \mathcal{O} , its logarithm, and the spectral polynomial

In this appendix we calculate the logarithm of \mathcal{O} , and its product with the projector $\hat{\mathbb{P}}$. We begin by representing \mathcal{O} in terms of a matrix B that squares to a matrix proportional to $\hat{\mathbb{P}}$. Defining

$$B \equiv \begin{pmatrix} 0 & -\frac{\bar{D}^2}{4\square} \frac{A_1^*(\square)}{A_2(\square)} \\ -\frac{D^2}{4\square} \frac{A_1(\square)}{A_2(\square)} & 0 \end{pmatrix}, \quad (\text{B1})$$

we have

$$B \hat{\mathbb{P}} = \hat{\mathbb{P}} B, \quad (\text{B2})$$

$$\text{tr}_{2 \times 2}(B) = \text{tr}_{2 \times 2}(\hat{\mathbb{P}} B) = 0, \quad (\text{B3})$$

$$\mathcal{O} = A_2(1 + B), \quad (\text{B4})$$

and

$$B^2 = \frac{|A_1|^2}{\square A_2^2} \hat{\mathbb{P}}, \quad (\text{B5})$$

where we have used the definitions (A5) and (A6) and the properties (A10). We will always be interested in tracing $\ln(\mathcal{O})$ or its product with $\hat{\mathbb{P}}$ as a 2×2 matrix, so odd powers of B will never be of interest to us, as they have zeroes on the diagonal. Thus we can write

$$\begin{aligned} \ln(1 + B) &= (\text{traceless } 2 \times 2) + \frac{1}{2} \ln(1 - B^2) \\ &= (\text{traceless } 2 \times 2) + \frac{1}{2} \ln\left(1 - \frac{|A_1|^2}{\square A_2^2} \hat{\mathbb{P}}\right) \\ &= (\text{traceless } 2 \times 2) + \frac{1}{2} \hat{\mathbb{P}} \ln\left(1 - \frac{|A_1|^2}{\square A_2^2}\right), \end{aligned} \quad (\text{B6})$$

and

$$\hat{\mathbb{P}} \ln(1 + B) = (\text{traceless } 2 \times 2) + \frac{1}{2} \hat{\mathbb{P}} \ln\left(1 - \frac{|A_1|^2}{\square A_2^2}\right), \quad (\text{B7})$$

so

$$\ln(\mathcal{O}) = \ln(A_2) \mathbf{1} + \frac{1}{2} \hat{\mathbb{P}} \ln\left(1 - \frac{|A_1|^2}{\square A_2^2}\right) + (\text{traceless } 2 \times 2) , \quad (\text{B8})$$

and

$$\begin{aligned} \hat{\mathbb{P}} \ln(\mathcal{O}) &= \hat{\mathbb{P}} \left[\ln(A_2) + \frac{1}{2} \ln\left(1 - \frac{|A_1|^2}{\square A_2^2}\right) \right] + (\text{traceless } 2 \times 2) \\ &= \frac{1}{2} \hat{\mathbb{P}} \ln \left(A_2^2 - \frac{|A_1|^2}{\square} \right) + (\text{traceless } 2 \times 2) \end{aligned} \quad (\text{B9})$$

Together with (A12), we use this identity to derive formula (3.18).

We also use the matrix B to compute the logarithm of the spectral polynomial $\sigma(P^2)$. This is given by

$$\begin{aligned} \ln(\sigma(P^2)) &= \text{Ctr} \left(\ln(\sqrt{\square} \mathcal{O}) \right) = \text{Ctr} \left(\ln(\sqrt{\square} A_2(\square)) \cdot \mathbf{1} \right) + \text{Ctr} \left(\ln(\mathbf{1} + B) \right) \\ &= \text{tr}(\hat{\mathbb{P}}) \cdot \left(\ln(\sqrt{\square} \cdot A_2(\square)) + \frac{1}{2} \ln\left(1 - \frac{|A_1(\square)|^2}{\square A_2(\square)^2}\right) \right) \\ &= \frac{1}{2} \text{tr}(\hat{\mathbb{P}}) \cdot \ln \left(\square \cdot (A_2(\square)^2 - \frac{|A_1(\square)|^2}{\square}) \right) . \end{aligned} \quad (\text{B10})$$

Now, $\text{tr}(\hat{\mathbb{P}}) = \text{Ctr}(\mathbf{1}) = 8$, so

$$\ln(\sigma(P^2)) = 4 \ln \left(\square \cdot (A_2(\square)^2 - \frac{|A_1(\square)|^2}{\square}) \right) , \quad (\text{B11})$$

and thus

$$\sigma(P^2) = \square^4 \left(A_2(\square)^4 - \frac{|A_1(\square)|^2}{\square} \right)^4 \quad (\text{B12})$$

For formula (4.11), we also need that

$$\begin{aligned} \frac{1}{4\square} \ln \text{Cdet}_{E_P}(\mathcal{O}) &= \frac{1}{4\square} \ln \left(\frac{\text{Cdet}(\sqrt{\square} \mathcal{O})}{\text{Cdet}(\sqrt{\square} \cdot \mathbf{1})} \right) \\ &= \frac{1}{4\square} \left(\ln(\sigma(P^2)) - \ln \text{Cdet}(\sqrt{\square} \cdot \mathbf{1}) \right) \\ &= \frac{1}{4\square} \ln(\sigma(P^2)) - \frac{1}{4\square} \ln \left((\sqrt{\square})^8 \right) . \end{aligned} \quad (\text{B13})$$

In section 4, we want to perform a similar computation in the case where we have several chiral superfields. It is useful to use the following identities for determinants of block matrices:

$$\begin{aligned} \det \begin{pmatrix} A & c \\ b & D \end{pmatrix} &= \det(D) \det(A - c D^{-1} b) \\ &= \det(A) \det(D) \det(1 - A^{-1} c D^{-1} b) . \end{aligned} \quad (\text{B14})$$

In our case we have $A = D^* = A_2(\square)$, $b = c^* = -\frac{D^2}{4\square} A_1(\square)$. Then for constant, supersymmetric backgrounds the operators D, \bar{D} and \square all commute through A_1, A_2 and their conjugates, so

$$\det(\mathcal{O}) = |\det(A_2)|^2 \det \left(1 - A_2^{-1} A_1^* A_2^{*-1} A_1 \frac{P_{\chi}}{\square} \right). \quad (\text{B15})$$

This is not yet quite the object we want; we really want a constrained determinant rather than a full determinant. We have

$$\text{Cdet} \begin{pmatrix} A & c \\ b & D \end{pmatrix} = \text{Cdet} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \text{Cdet}(1 + K), \quad (\text{B16})$$

where

$$K \equiv \begin{pmatrix} 0 & A^{-1}c \\ D^{-1}b & 0 \end{pmatrix}. \quad (\text{B17})$$

For any $a \geq 1$, the matrix K has the properties that $K^a = \hat{\mathbb{P}} K^a = K^a \hat{\mathbb{P}}$ and $\text{Ctr}(K^{2a-1}) = 0$. For any matrix with these properties,

$$\text{Cdet}(1 + K) = \sqrt{\det(1 - K^2)}. \quad (\text{B18})$$

Here,

$$K^2 = \begin{pmatrix} A^{-1}cD^{-1}b & 0 \\ 0 & D^{-1}bA^{-1}c \end{pmatrix}. \quad (\text{B19})$$

For us, (4.5) leads to

$$K^2 = \begin{pmatrix} A_2^{-1} A_1^* A_2^{*-1} A_1 \frac{P_{\chi}}{\square} & 0 \\ 0 & A_2^{*-1} A_1^* A_2^{-1} A_1 \frac{P_{\bar{\chi}}}{\square} \end{pmatrix}. \quad (\text{B20})$$

Then

$$\det(1 - K^2) = (\det_{n \times n}(1 - A_2^{-1} A_1^* A_2^{*-1} A_1 \square^{-1}))^8 \quad (\text{B21})$$

and

$$\begin{aligned} \text{Cdet}(1 + K) &= (\det_{n \times n}(1 - A_2^{-1} A_1^* A_2^{*-1} A_1 \square^{-1}))^4 \\ &= (\det_{n \times n}(1 - A_2^{*-1} A_1 A_2^{-1} A_1^* \square^{-1}))^4 \end{aligned} \quad (\text{B22})$$

and

$$\text{Cdet} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \text{Cdet} \begin{pmatrix} A_2 & 0 \\ 0 & A_2^* \end{pmatrix} = (\det_{n \times n}(A_2))^4 (\det_{n \times n}(A_2^*))^4, \quad (\text{B23})$$

so

$$\begin{aligned} \text{Cdet}(\mathcal{O}) &= |\det_{n \times n}(A_2)|^8 (\det_{n \times n}(1 - A_2^{*-1} A_1 A_2^{-1} A_1^* \square^{-1}))^4 \\ &= (\det_{n \times n}(A_2 A_2^* - A_1 A_2^{-1} A_1^* A_2 \square^{-1}))^4. \end{aligned} \quad (\text{B24})$$

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